

Generalized Soft Wall Model

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Abstract

We develop an exactly solvable generalization of the soft wall holographic model for the vector mesons. The generalization preserves the ultraviolet and infrared asymptotics of the soft wall model and contains an additional free parameter. This parameter provides an arbitrary intercept in the Regge like spectrum of radial excitations and leads to a substantial modification of asymptotic expansion of the vector correlator at large momentum. The matching to the Operator Product Expansion from QCD allows to make a strong restriction on possible values of the new parameter.

1 Introduction

One of spectacular manifestations of confinement in QCD is the expected Regge behavior of meson spectrum in the light quark sector, both in the spin and in the radial directions. Traditionally the related phenomenology was discussed in terms of the effective strings or dual amplitudes. The story has taken an interesting turn with the appearance of the Soft Wall (SW) holographic model for the strongly coupled QCD [1] which was inspired by the ideas of the gauge/gravity correspondence in the string theory [2–4]. Since the SW model provided a seminal theoretical setup for many applications (a uncomplete list of the most recent developments is given in [5]), a natural question arises whether it is possible to generalize it. Unfortunately, the existing extensions and modifications of this model (at least which we know) spoil its main attractive feature consisting in the simplicity and exact solvability. In the present paper, we propose a simple generalization of the SW model which preserves its properties and is exactly solvable. In the framework of our generalization, the intercept of the linear mass spectrum is not fixed as in the original model [1], instead it can take an arbitrary value. This feature modifies the structure of the vector correlator. The matching to the operator product expansion allows to impose hard restrictions on the intercept which follow from QCD.

The paper is organized as follows. The scheme of the vector SW model is briefly reminded in Sect. 2. In Sect. 3 we derive our generalization of this model. The two-point correlator and its high-momentum expansion are calculated in Sect. 4, where some fits are also given. The final Sect. 5 contains some phenomenological discussions and our conclusions.

2 Soft Wall Model

In this section, we remind the reader the basic aspects of the SW holographic model [1]. For the sake of simplicity we will consider the simplest Abelian version of this model that is defined by the 5D action

$$S = \int d^4x dz \sqrt{g} e^{-az^2} \left(-\frac{1}{4g_5^2} F_{MN} F^{MN} \right), \quad (1)$$

where $g = |\det g_{MN}|$, $F_{MN} = \partial_M V_N - \partial_N V_M$, $M, N = 0, 1, 2, 3, 4$, in the AdS background space whose metrics can be parametrized as

$$g_{MN} dx^M dx^N = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2). \quad (2)$$

Here $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, R denotes the AdS radius, and $z > 0$ is the holographic coordinate having the physical sense of inverse energy scale. The boundary $z = 0$ represents the 4D Minkowski space.

The gauge invariance of the action (1) permits to choose the axial gauge, $V_z = 0$, in which the equation of motion is simplified. The spectrum m_n of the physical vector mesons emerges from the Kaluza-Klein decomposition of the field V_μ ,

$$V_\mu(x, z) = \sum_{n=0}^{\infty} V_\mu^{(n)}(x) v_n(z). \quad (3)$$

For calculating the mass spectrum one should find the normalizable solutions of the equation of motion for the 4D Fourier transform $V_\mu^T(q, z)$ of the transverse components ($\partial^\mu V_\mu^T = 0$). The normalizable eigenfunctions $v_n(z)$ exist only for discrete values of 4D momentum $q_n^2 = m_n^2$. The corresponding equation that follows from the action (1) reads as follows

$$\partial_z \left(\frac{e^{-az^2}}{z} \partial_z v_n \right) + m_n^2 \frac{e^{-az^2}}{z} v_n = 0. \quad (4)$$

This is a typical Sturm-Liouville problem. It is convenient to make the substitution

$$v_n = \sqrt{z} e^{az^2/2} \psi_n, \quad (5)$$

which transforms the Eq. (4) in a Schrödinger equation

$$-\psi_n'' + V(z)\psi_n = m_n^2\psi_n, \quad (6)$$

$$V(z) = a^2 z^2 + \frac{3}{4z^2}, \quad (7)$$

where the prime stays for ∂_z . The eigenvalues of Eq. (6) yield the mass spectrum of the model

$$m_n^2 = 4|a|(n+1). \quad (8)$$

Although the spectrum (8) does not depend on the sign of a , the choice $a < 0$ leads to unphysical zero mode [6]. For this reason we will assume $a > 0$ in what follows.

In a more general situation (for other spins) the potential (7) has two additional parameters

$$V(z) = a^2 z^2 + \frac{m^2 - 1/4}{z^2} + 4ab, \quad (9)$$

and results in the spectrum

$$m_n^2 = 2a(2n + m + 1 + 2b). \quad (10)$$

3 Generalized Soft Wall Model

We wish to derive an exactly solvable generalization of the vector SW model that has an arbitrary intercept in the mass spectrum,

$$m_n^2 = 4a(n + 1 + b). \quad (11)$$

Our generalization must not spoil neither ultraviolet (UV) nor infrared (IR) asymptotics of the original SW model. We are going to show that this requirement fixes unambiguously the form of the background in the 5D action.

Let us write the holographic action in the form

$$S = \int d^4x dz f^2 \left(-\frac{1}{4g_5^2} F_{MN}^2 \right), \quad (12)$$

with the unknown function $f(z)$ to be determined. The conformal symmetry dictates the following UV asymptotics for this function,

$$f(z)|_{z \rightarrow 0} \sim \frac{1}{\sqrt{z}}. \quad (13)$$

If the condition (13) is satisfied then in the UV limit the action (12) (written in the covariant form) has a form of the action (1).

The equation for the mass spectrum is

$$(f^2 v_n')' + f^2 m_n^2 v_n = 0. \quad (14)$$

The substitution

$$v_n = \frac{\psi_n}{f}, \quad (15)$$

brings the Eq. (14) into the form of a Schrödinger equation

$$-\psi_n'' + \frac{f''}{f} \psi_n = m_n^2 \psi_n. \quad (16)$$

From (10) it follows that for obtaining a shift in the intercept the potential $\frac{f''}{f}$ must have the form of (9). However, the choice $m^2 \neq 1$ will lead to a wrong UV asymptotics in the vector SW model. The only possibility is to find the function f from the condition

$$\frac{f''}{f} = a^2 z^2 + \frac{3}{4z^2} + 4ab, \quad (17)$$

which has the form of Eq. (6) with m_n^2 replaced by $-4ab$. This condition ensures the spectrum (11) we are looking for.

The equation (17) has two solutions — an exponentially decreasing and an exponentially growing one. To comply with the IR asymptotics of the SW model (dictated by the absence of massless mode and by the correct spectrum for the higher spin mesons [6]) we must select the decreasing solution. Neglecting also the cases $b = -1, -2, \dots$ (since we do not want to have any massless or tachyonic modes) the corresponding solution is

$$f = \text{const} \cdot \frac{e^{-az^2/2}}{\sqrt{z}} U(b, 0; az^2), \quad (18)$$

where U is the Tricomi confluent hypergeometric function.

Thus the action of the generalized SW model reads

$$S = \int d^4x dz \sqrt{g} e^{-az^2} U^2(b, 0; az^2) \left(-\frac{1}{4g_5^2} F_{MN} F^{MN} \right). \quad (19)$$

This is our main result. Since $U(b, 0; 0) = \text{const}$ and $U(b, 0; az^2) \rightarrow (az^2)^{-b}$ at $z \rightarrow \infty$, the obtained modification of the 5D background does not affect neither UV nor leading IR asymptotics. If for some reason one considers the SW model with inverse dilaton background, $a < 0$, the argument of the Tricomi function must be changed to $|a|z^2$ (the function U is complex at negative argument).

4 Vector Correlator

The introduction of arbitrary shift in the spectrum brings qualitatively new properties to structure of the correlation functions. Following the standard recipe for the holographic calculation of the correlators [3, 4], first we should find the solution $v(q, z)$ (q is the 4D momentum) of equation of motion which is subject to the boundary condition $v(q, 0) = 1$. For the action (19) the corresponding solution is

$$v(q, z) = \frac{\Gamma(1 + b - \tilde{q}^2) U(b - \tilde{q}^2, 0; az^2)}{\Gamma(1 + b) U(b, 0; az^2)}, \quad (20)$$

where the dimensionless momentum has been introduced, $\tilde{q}^2 \equiv \frac{q^2}{4a}$. The two-point correlation function of vector currents J_μ ,

$$\int d^4x e^{iqx} \langle J_\mu(x) J_\nu(0) \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi_V(q^2), \quad (21)$$

can be expressed via $v(q, z)$ [7, 8],

$$\Pi_V(q^2) = \frac{R}{g_5^2} \frac{\partial_z v}{q^2 z} \Big|_{z \rightarrow 0}. \quad (22)$$

The substitution of (20) to (22) gives the expression

$$\Pi_V(q^2) = \frac{R}{2g_5^2} \left\{ -\psi(1 + b - \tilde{q}^2) + \frac{b}{\tilde{q}^2} [\psi(1 + b - \tilde{q}^2) - \psi(1 + b)] \right\}, \quad (23)$$

where ψ denotes the digamma function. Applying the decomposition

$$\psi(1 + x) = -\sum_{n=0}^{\infty} \frac{1}{x + n + 1} + \text{const}, \quad (24)$$

we arrive at the spectral representation for the correlator under consideration,

$$\Pi_V(q^2) = -\sum_{n=0}^{\infty} \frac{F_n^2}{q^2 - 4a(n + 1 + b)}, \quad (25)$$

$$F_n^2 = \frac{2aR}{g_5^2} \left(1 - \frac{b}{4a(n + 1 + b)} \right). \quad (26)$$

The poles of the correlator yield the mass spectrum (11). At $b \neq 0$ the residues (they determine the electromagnetic decay width, $\Gamma_{\text{em}} \sim F^2$) acquire a dependence on n . Choosing $b < 0$ this new feature allows to mimic a kind

of the vector dominance: The lightest vector state possesses the largest value of residue.

The expansion of the correlator (23) at large Euclidean momentum $Q^2 = -q^2$ leads to

$$\begin{aligned} \Pi_V|_{Q^2 \rightarrow \infty} = & \frac{R}{2g_5^2} \left\{ \log \left(\frac{4a}{Q^2} \right) - \frac{4a}{Q^2} \left[\frac{1}{2} + b \left(\log \left(\frac{4a}{Q^2} \right) + 1 - \psi(1+b) \right) \right] \right. \\ & \left. + \frac{1}{2} \left(\frac{4a}{Q^2} \right)^2 \left(\frac{1}{6} - b^2 \right) + \frac{1}{6} \left(\frac{4a}{Q^2} \right)^3 b \left(b^2 - \frac{1}{2} \right) + \mathcal{O}(Q^{-8}) \right\}. \end{aligned} \quad (27)$$

The expansion (27) can be matched to the Operator Product Expansion (OPE) for the vector two-point correlator [9],

$$\Pi_V^{(\text{OPE})} = \frac{N_c}{24\pi^2} \log \left(\frac{\mu_{\text{ren}}^2}{Q^2} \right) + \frac{\alpha_s}{24\pi} \frac{\langle G^2 \rangle}{Q^4} - \frac{14\pi\alpha_s}{9} \frac{\langle \bar{q}q \rangle^2}{Q^6} + \mathcal{O}(Q^{-8}). \quad (28)$$

The matching of coefficients in front of the leading logarithm provides the standard normalization factor,

$$\frac{R}{g_5^2} = \frac{N_c}{12\pi^2}. \quad (29)$$

In principle, the free parameters of the model — the slope $4a$ and the (dimensionless) contribution to the intercept b — can be fixed by matching the $\mathcal{O}(Q^{-4})$ and $\mathcal{O}(Q^{-6})$ terms. For the typical phenomenological values of the gluon and quark condensates, $\frac{\alpha_s}{\pi} \langle G^2 \rangle = (360 \text{ MeV})^4$ and $\langle \bar{q}q \rangle = -(235 \text{ MeV})^3$, one obtains $4a = (905 \text{ MeV})^2$ and $b = 0.046$. Taking into account the qualitative character of the model, these estimates look reasonable.

A more conservative point of view on the $\mathcal{O}(Q^{-6})$ term would be to consider it as non-reliable for numerical fits because of the asymptotic nature of the expansion. Within the standard SW model, $b = 0$, taking a typical phenomenological value for the slope of meson trajectories [10], $4a \approx (1.1 \text{ GeV})^2$, the matching of $\mathcal{O}(Q^{-4})$ terms in the expansions (27) and (28) predicts a unrealistically large value for the gluon condensate, $\frac{\alpha_s}{\pi} \langle G^2 \rangle \approx (440 \text{ MeV})^4$. The parameter b allows to remove this drawback: It can be fixed from the condition to have a realistic gluon condensate in the expansions (27),

$$b^2 = \frac{1}{6} - \frac{2\pi^2 \frac{\alpha_s}{\pi} \langle G^2 \rangle}{N_c (4a)^2}. \quad (30)$$

Substituting the physical values for the slope and gluon condensate to the condition (30), we arrive at the estimate $|b| \approx 0.3$.

5 Discussions and Conclusions

The vector correlator of the SW model contains the $\mathcal{O}(Q^{-2})$ term in the expansion at large Euclidean momentum. Such a term is absent in the OPE (28) by virtue of the absence of dim2 local gauge-invariant operator in QCD (although there are many speculations about the phenomenological relevance of dim2 condensate [11]). Unfortunately, the generalized SW model cannot solve this problem because of the logarithm in the numerator of $\mathcal{O}(Q^{-2})$ term in the expansion (27). More precisely, the problem can be partly resolved if one eliminates the constant part in this numerator by fine-tuning the parameter b . A possible physical origin of the residual $\frac{\log Q^2}{Q^2}$ term in the OPE remains however unclear.

The positivity of the $\mathcal{O}(Q^{-4})$ term in the OPE leads to the constraint $|b| < 1/\sqrt{6}$ in the expansion (27). In addition, since this term is universal in the OPE for the vector and axial-vector channels [9] and depends quadratically on b , one has an intriguing possibility for a self-consistent mass splitting between the vector (V) and axial (A) states: The corresponding spectra have universal absolute value of b but opposite sign,

$$m_{V,A}^2(n) = 4a(n + 1 \mp |b|). \quad (31)$$

The nonzero values of b generate the $\mathcal{O}(Q^{-6})$ term in the OPE and this represents a new feature in comparison with the usual SW model. In the latter, the term $\mathcal{O}(Q^{-6})$ is absent because the intercept (in units of the slope) is equal to unity. It is well known [12] that this is one of values of intercept at which the term $\mathcal{O}(Q^{-6})$ disappears in the OPE of the two-point correlators saturated by the narrow resonances with linearly rising spectrum. It is interesting to note that in the model (31), the $\mathcal{O}(Q^{-6})$ term in the V and A correlators differ by sign only (see Eq. (27)). This is somewhat close to the real OPE where these terms differ by the factor $-\frac{7}{11}$ [9].

The extension of the SW model to the higher spin (S) mesons leads to a nice relation $m_{n,S}^2 = 4a(n + S)$ [1] which is compatible with other approaches (the Nambu-Goto strings, Veneziano amplitudes). The background in the action (19) does not lead to a simple shift in the spectrum for higher S (for the latter purpose one would need a different background for each spin). One can show that introducing the higher spin states according to the scheme of Ref. [1], the potential (17) of a Schrödinger equation in the background (19) becomes

$$V(z) = a^2 z^2 + 2a(S - 1) + \frac{S^2 - 1/4}{z^2} + 4ab[1 + (S - 1)\zeta(z)], \quad (32)$$

where the function $\zeta(z)$,

$$\zeta(z) = \frac{U(1+b, 1; az^2)}{U(b, 0; az^2)}, \quad (33)$$

behaves as $\zeta(z \rightarrow 0) \sim -\log z$ and $\zeta(z \rightarrow \infty) \sim z^{-2}$ in the UV and IR limits. Consequently the contribution due to nonzero b does not affect the UV and IR asymptotics of the potential (32). This contribution will cause a slight deviation from linearity of the Regge like spectrum.

In summary, we have shown how to introduce an arbitrary intercept to the linear spectrum of the SW model in a self-consistent way. The obtained generalization of the SW model remains exactly solvable. The resulting freedom in the choice of the intercept entails a sizeable modification of the two-point correlator, specifically a contribution related to the quark condensate is generated and the residues of meson poles cease to be universal for all states. In view of recent phenomenological applications of the SW model and attempts to derive it from a more fundamental setup, its generalization presented in this work may be useful.

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References

- [1] A. Karch, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. D **74**, 015005 (2006).
- [2] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
- [3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
- [4] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
- [5] Y. Y. Bu, Phys. Rev. D **86**, 026003 (2012); S. Liu and P. Zhang, Phys. Rev. D **86**, 014015 (2012); T. Gutsche, V. E. Lyubovitskij, I. Schmidt and A. Vega, Phys. Rev. D **86**, 036007 (2012); J. R. Forshaw and R. Sandapen, Phys. Rev. Lett. **109**, 081601 (2012); L. -X. Cui, S. Takeuchi

- and Y. -L. Wu, JHEP **1204**, 144 (2012); P. Colangelo, F. Giannuzzi, S. Nicotri and V. Tangorra, Eur. Phys. J. C **72**, 2096 (2012).
- [6] A. Karch, E. Katz, D. T. Son and M. A. Stephanov, JHEP **1104**, 066 (2011).
 - [7] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. **95**, 261602 (2005).
 - [8] L. Da Rold and A. Pomarol, Nucl. Phys. B **721**, 79 (2005).
 - [9] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **147**, 385, 448 (1979).
 - [10] A. V. Anisovich, V. V. Anisovich and A. V. Sarantsev, Phys. Rev. D **62**, 051502(R) (2000); D. V. Bugg, Phys. Rept. **397**, 257 (2004).
 - [11] K. G. Chetirkin, S. Narison, and V. I. Zakharov, Nucl. Phys. B **550**, 353 (1999); F. V. Gubarev and V. I. Zakharov, Phys. Lett. B **501**, 28 (2001); K.-I. Kondo, Phys. Lett. B **514**, 335 (2001); H. Verschelde, K. Knecht, K. Van Acoleyen, and M. Vanderkelen, Phys. Lett. B **516**, 307 (2001); P. Boucaud et al., Phys. Rev. D **63**, 114003 (2001); E. Ruiz Arriola, P. O. Bowman, and W. Broniowski, Phys. Rev. D **70**, 097505 (2004); E. Ruiz Arriola and W. Broniowski, Phys. Rev. D **81**, 054009 (2010).
 - [12] S. S. Afonin, Phys. Lett. B **576**, 122 (2003); S. S. Afonin, A. A. Andrianov, V. A. Andrianov and D. Espriu, JHEP **0404**, 039 (2004); S. S. Afonin and D. Espriu, JHEP **0609**, 047 (2006).